JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 3, August 2012

THE TOEPLITZ OPERATOR INDUCED BY AN R-LATTICE

Si Ho Kang*

Abstract. The hyperbolic metric is invariant under the action of Möbius maps and unbounded. For $0 < r < 1$, there is an r-lattice in the Bergman metric. Using this r-lattice, we get the measure μ_r and the Toeplitz operator $T^{\alpha}_{\mu_r}$ and we prove that $T^{\alpha}_{\mu_r}$ is bounded and $T^{\alpha}_{\mu_r}$ is compact under some condition.

1. Introduction

Let $dA = \dfrac{1}{\pi}dxdy$ denote the normalized Lebesgue area measure on the unit disk and for $\alpha > -1$, let $dA_{\alpha}(z) = (\alpha+1)(1-|z|^2)^{\alpha}dA(z)$. For $p \geq 1$, the weighted Bergman space A^p_{α} consists of the analytic functions in $L^p(\mathbb{D}, dA_{\alpha})$. Then A^2_{α} is closed in $L^2(\mathbb{D}, dA_{\alpha})$ and for each $z \in \mathbb{D}$, there is a reproducing kernel K^{α}_z in A^2_{α} such that $f(z) =< f, K^{\alpha}_z >$ for all $f \in A^2_{\alpha}$, where $K^{\alpha}_z(w) = \dfrac{1}{(1-\overline{z}w)^{2+\alpha}}$. Moreover, we get the orthogonal projection P_{α} from $L^2(\mathbb{D}, dA_{\alpha})$ onto A^2_{α} defined by $P_{\alpha}(f)(z) =< f, K^{\alpha}_z >$ for all $f \in L^2(\mathbb{D}, dA_{\alpha})$, where the norm $||\cdot||_{2,\alpha}$ and the inner product are taken in the space $L^2(\mathbb{D}, dA_{\alpha})$.

Let $\mathrm{Aut}(\mathbb{D})$ be the set of all Möbius maps of $\mathbb{D}$, that is, each element of $\mathrm{Aut}(\mathbb{D})$ is bijective and analytic. By Schwarz lemma, φ is a Möbius map if and only if there is a unit modulus constant λ and a point $z \in \mathbb{D}$ such that $\varphi(w) = \lambda\varphi_z(w)$, $w \in \mathbb{D}$, where $\varphi_z(w) = \dfrac{z-w}{1-\overline{z}w}$ is a linear fractional transformation.

Received March 19, 2012; Accepted June 20, 2012.
2010 Mathematics Subject Classification: Primary 47A15, 47B47.
Key words and phrases: weighted Bergman spaces, Toeplitz operators, r-lattice, Carleson measure.
*This research was partially supported by Sookmyung Women's University Research Grants 2012.

For a complex measure μ on $\mathbb{D}$ and $u \in A^2_\alpha$, we define the Toeplitz operator with symbol μ by

$$T^\alpha_\mu(u)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha u(w)}{(1-\overline{w}z)^{2+\alpha}} d\mu(w).$$

For $f \in L^1(\mathbb{D}, dA)$, $d\mu(z) = f(z)dA(z)$ is a measure with the Radon-Nikodym derivative f. Then

$$\begin{aligned} T^\alpha_\mu(u)(z) &= (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha u(w)}{(1-\overline{w}z)^{2+\alpha}} f(w)dA(w) \\ &= T^\alpha_f(u)(z), \end{aligned}$$

where T^α_f is the Toeplitz operator with symbol f. Since $L^\infty(\mathbb{D}, dA)$ is dense in $L^1(\mathbb{D}, dA)$, T^α_f is densely defined on A^2_α. In fact, $T^\alpha_\mu(u) = T^\alpha_f(u) = P_\alpha(uf)$.

By Fubini's theorem, for two polynomials f and g,

$$\begin{aligned} < T^\alpha_\mu f, g > &= \int_{\mathbb{D}} T^\alpha_\mu f(z)\overline{g(z)} dA_\alpha(z) \\ &= \int_{\mathbb{D}} (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha f(w)}{(1-\overline{w}z)^{2+\alpha}} d\mu(w)\overline{g(z)} dA_\alpha(z) \\ &= (\alpha+1)\int_{\mathbb{D}} (1-|w|^2)^\alpha f(w)\overline{\int_{\mathbb{D}} \frac{g(z)}{(1-\overline{w}z)^{2+\alpha}} dA_\alpha(z)} d\mu(w) \\ &= (\alpha+1)\int_{\mathbb{D}} (1-|w|^2)^\alpha f(w)\overline{g(w)} d\mu(w). \end{aligned}$$

Since the set of all polynomials is dense in $C(\overline{\mathbb{D}})$, $T^\alpha_\mu = 0$ if and only if $\mu = 0$.

On the other hand, since $P_\gamma f(z) = (\gamma+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma f(w)}{(1-z\overline{w})^{2+\gamma}} dA(w)$ $= \frac{\gamma+1}{\alpha+1}\int_{\mathbb{D}} (\alpha+1) \frac{(1-|w|^2)^\alpha (1-|w|^2)^{\gamma-\alpha}}{(1-z\overline{w})^{2+\alpha+(\gamma-\alpha)}} f(w)dA(w)$, whenever $\alpha \le \gamma$, $P_\gamma(L^1(\mathbb{D}, dA_\alpha)) \subseteq A^1_\alpha$ and $P_\gamma(H(\mathbb{D})) \subseteq H(\mathbb{D})$.

In the next section, we introduce the measure induced by an r-lattice and show that the Toeplitz operator induced by the measure is bounded and converges to the identity operator as $r \to 0$. Moreover, we show that the Toeplitz operator is compact under some condition. Throughout this paper, we use the symbol $A \preceq B$ ($A \approx B$, resp.) for nonnegative constants A and B to indicate that A is dominated by B times some positive constant ($A \preceq B$ and $B \preceq A$, resp.).

2. The main theorems

Let μ be a finite positive Borel measure on $\mathbb{D}$ and $p > 0$. We say that μ is a Carleson measure for the Bergman space A^p_α if there exists a finite constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \quad \leq \quad C \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)$$

for all $f \in A^p_\alpha$. The Closed Graph Theorem shows that A^p_α is contained in $L^p(\mathbb{D}, d\mu)$ if and only if the inclusion map $i_p : A^p_\alpha \to L^p(\mathbb{D}, d\mu)$ is bounded. By the above observation, for any two polynomials f and g,

$$\begin{aligned} |< T^\alpha_\mu f, g >| \quad &\leq \quad (\alpha+1) \int_{\mathbb{D}} |f(w)\overline{g(w)}| d\mu(w) \\ &\leq \quad C \int_{\mathbb{D}} |f(w)\overline{g(w)}| dA_\alpha(w) \\ &\leq \quad C||f||_{2,\alpha}||g||_{2,\alpha}. \end{aligned}$$

Here the second inequality follows from the fact tha μ is a Carleson measure. Thus T^α_μ is bounded on A^2_α.

The following is Theorem 7.4 in Zhu[4].

Theorem 2.1. *Suppose μ is a finite positive Borel measure on $\mathbb{D}$, $p > 0$, $\alpha > -1$ and $r > 0$. Then μ is a Carleson measure for A^p_α if and only if* $\sup \left\{ \frac{\mu(D(a,r))}{(1-|a|^2)^{2+\alpha}} : a \in \mathbb{D} \right\} < +\infty$. *Here $D(a, r)$ is a Bergman disk.*

Suppose $0 < r < 1$. Then there exists an r-lattice $\{a_n\}$ in the Bergman metric such that for each k there exists a measurable set D_k with the following propertices :

(a) $D(a_k, \frac{r}{4}) \subseteq D_k \subseteq D(a_k, r)$ for all $k \geq 1$.

(b) $D_n \cap D_m = \phi$ if $m \neq n$.

(c) $D_1 \cup D_2 \cup \cdots = \mathbb{D}$.

(d) Every point of $\mathbb{D}$ belongs to at most $c\frac{\sigma^2}{r^2}$ of the sets $\Omega_\sigma(D_j) = \{z : \beta(z, D_j) \leq \sigma\}$.

We define $\mu_r = \sum_{m=1}^{\infty} |D_m|\delta_{\{a_m\}}$, where $|D|$ is the area of any measurable set D in $\mathbb{D}$ with respect to the normalized area measure dA. Then μ_r is a positive measure.

Suppose $f \in A^2_\alpha$ and $z \in \mathbb{D}$. Then

$$\begin{aligned} f(z) &= P_\alpha f(z) = < f, K^\alpha_z > \\ &= \int_{\mathbb{D}} f(w) \frac{1}{(1 - z\overline{w})^{2+\alpha}} dA_\alpha(w) \\ &= (\alpha+1) \sum_m \int_{D_m} f(w) \frac{(1-|w|^2)^\alpha}{(1-z\overline{w})^{2+\alpha}} dA(w) \end{aligned}$$

and

$$\begin{aligned} T^\alpha_{\mu_r}(f)(z) &= (\alpha+1) \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{(1-z\overline{w})^{2+\alpha}} f(w) d\mu_r(w) \\ &= (\alpha+1) \sum_m \frac{(1-|a_m|^2)^\alpha}{(1-z\overline{a_m})^{2+\alpha}} f(a_m)|D_m|. \end{aligned}$$

Since $T^\alpha_{\mu_r}(f)(z) = (\alpha+1)\sum_m f(a_m)\overline{k^\alpha_z(a_m)}|D_m|$, this series can be considered as an approximating Riemann sum of $P_\alpha(f)(z)$ which is closed to the actual integral.

THEOREM 2.2. *μ_r is a Carleson measure and hence $T^\alpha_{\mu_r}$ is bounded (see [1],[2]).*

Proof. Suppose $a_m \in D(z,1)$ and $w \in D_m$. Since $D_m \subseteq D(a_m, r)$, $\beta(w,z) \le \beta(w,a_m) + \beta(a_m,z) \le r+1 \le 2$ and hence

$$\begin{aligned} \mu_r(D(z,1)) &= \sum_{a_m \in D(z,1)} |D_m| = \sum_{a_m \in D(z,1)} \int_{D_m} dA(w) \\ &\le \sum_{D_m \subseteq D(z,2)} \int_{D_m} dA(w) \le \int_{D(z,2)} dA(w) \\ &= C|D(z,1)| \text{ for some constant C} \\ &\approx |D(z,1)|^{2+\alpha}. \end{aligned}$$

Thus μ_r is a Carleson measure. $\square$

Since $D(z,r) = \{w \in \mathbb{D} : \beta(w,z) < r\} = B\Big(\frac{1-s^2}{1-s^2|z|^2}z, \frac{1-|z|^2}{1-s^2|z|^2}s\Big)$, where $B(a,r)$ is a Euclidean disk with center a and radius r and $s = \tanh r$, $\lim_{r\to 0} |D_m|\delta_{\{a_m\}} = \lim_{r \to 0}|D_m|$ and hence we conjecture $\lim_{r\to 0} ||T^\alpha_{\mu_r} - I|| = 0$.

We notice that $\beta(w,z) < r$ if and only if $\rho(z,w) = \left|\frac{z-w}{1-z\overline{w}}\right| < \tanh(r) = s$. Since $|1 - z\overline{\varphi_z(w)}| = \dfrac{1-|z|^2}{|1-z\overline{w}|}$ for all $w \in D(0,r)$ and $1 - s|z| \le$

$|1 - z\overline{w}| \leq 1 + s|z|$, $(1-|w|^2)^{\alpha} \approx (1-|a_m|^2)^{\alpha} \approx |1 - a_m\overline{w}|^{\alpha}$ for $w \in D(a_m, r)$.

Using these observations, we get the following property :

Lemma 2.3. *$f \in A^2_{\alpha}$ and $0 < r < 0.51850$ then there exists a positive constant C_1 such that*

$$\int_{D(a_m,r)} |f(w) - f(a_m)|^2 dA_{\alpha}(w) \leq C_1 s^2 \int_{D(a_m, r^{\frac{1}{4}})} |f(w)|^2 dA_{\alpha}(w),$$

where $s =$tanh(r).

Proof. Since

$$\begin{aligned}
&\int_{D_m} |f(w) - f(a_m)|^2 dA_{\alpha}(w)\\
&\approx (\alpha+1)(1-|a_m|^2)^{\alpha} \int_{D_m} |f(w) - f(a_m)|^2 dA(w),
\end{aligned}$$

it is enough to show that it is true for $\int_{D(a_m,r)} |f(w) - f(a_m)|^2 dA(w)$. Since for $w \in D(a_m, r)$, $|K_{a_m}|^2 \approx \frac{1}{(1-|a_m|^2)^2}$,

$$\begin{aligned}
\int_{D_m} |f(w) - f(a_m)|^2 dA(w) &\leq \int_{D(a_m,r)} |f(w) - f(a_m)|^2 dA(w)\\
&= \int_{D(0,r)} |f \circ \varphi_{a_m}(w) - f \circ \varphi_{a_m}(0)|^2 |K_{a_m}(w)|^2 dA(w)\\
&\approx \frac{1}{(1-|a_m|^2)^2} \int_{D(0,r)} |f \circ \varphi_{a_m}(w) - f \circ \varphi_{a_m}(0)|^2 dA(w).
\end{aligned}$$

Let $F(w) = f \circ \varphi_{a_m}(w)$ and let $s =$tanh(r). Since F is analytic, put $F(w) = \sum_{n=0}^{\infty} b_n w^n$. Since $F(w) - F(0) = w \sum_{n=1}^{\infty} b_n w^{n-1}$ and $w \in D(0, r)$,

$$\begin{aligned}
|F(w) - F(0)|^2 &\leq |w|^2 \Big(\sum_{n=1}^{\infty} |b_n||w|^{n-1} \Big)^2\\
&= |w|^2 \Big[\sum_{n=1}^{\infty} \Big(\frac{|b_n|}{\sqrt{n+1}} |w|^{\frac{1}{2}(n-1)} \Big) \Big(\sqrt{n+1} |w|^{\frac{1}{2}(n-1)} \Big) \Big]^2\\
&\leq s^2 \Big(\sum_{n=1}^{\infty} \frac{|b_n|^2}{n+1} |w|^{n-1} \Big) \Big(\sum_{n=1}^{\infty} (n+1) s^{n-1} \Big).
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(n+1)s^{n-1}$ is convergent and the functions $y=$tanh(r) and $y=(\tanh(r^{\frac{1}{4}}))^2$ intersect at the point $(0,0)$ and $(0.51850, 0.47654)$, $F(w)-F(0)$ is in $L^2(\mathbb{D}, dA)$ and whenever $0<r<0.51850$, $s\leq(\overline{s})^2$ where $\overline{s}=$tanh$(r^{\frac{1}{4}})$. Since $0<s<1$,

$$\begin{aligned}\int_{D(0,r)}|F(w)-F(0)|^2dA(w) &= \int_{B(0,s)}\Big|\sum_{n=1}^{\infty}b_nw^n\Big|^2dA(w)\\ &= \int_0^s\sum_{n=1}^{\infty}|b_n|^2t^{2n+1}dt=\sum_{n=1}^{\infty}\frac{|b_n|^2}{2n+2}s^{2n+2}\\ &\leq s^2\sum_{n=0}^{\infty}\frac{|b_n|^2}{2n+2}s^{n+1}\end{aligned}$$

and

$$\begin{aligned}\int_{D(0,r^{\frac{1}{4}})}|F(w)|^2dA(w) &= \sum_{n=0}^{\infty}\int_0^{\overline{s}}|b_n|^2t^{2n+1}dt\\ &= \sum_{n=0}^{\infty}\frac{|b_n|^2}{2n+2}(\overline{s})^{2(n+1)}\geq\sum_{n=0}^{\infty}\frac{|b_n|^2}{2n+2}s^{n+1}.\end{aligned}$$

Thus

$$\int_{D(a_m,r)}|f(w)-f(a_m)|^2dA(w) \leq C_1s^2\int_{D(a_m,r^{\frac{1}{4}})}|f(w)|^2dA(w).$$

for some positive constant C_1. This implies the result. □

THEOREM 2.4. *Let I be the identity operator on A^2_α. Then* $\lim_{r\to 0}||I-T^\alpha_{\mu_r}||=0$.

Proof. Take any f and g in A^2_α. Then

$$\begin{aligned}&<(I-T^\alpha_{\mu_r})f,g>\\ &=<f,g>-<T^\alpha_{\mu_r}(f),g>\\ &=\sum_m\int_{D_m}f(w)\overline{g(w)}dA_\alpha(w)-\sum_m(\alpha+1)f(a_m)\overline{g(a_m)}|D_m|(1-|a_m|^2)^\alpha\end{aligned}$$

$$\approx \sum_m \int_{D_m} (f(w) - f(a_m))\overline{g(w)} dA_\alpha(w) + \sum_m \int_{D_m} f(a_m)\overline{(g(w) - g(a_m))} dA_\alpha(w).$$

Since $\Big(\sum_m \int_{D_m} (f(w) - f(a_m))\overline{g(w)} dA_\alpha(w)\Big)^2$

$$\leq \Big[\sum_m \Big(\int_{D_m} |f(w) - f(a_m)|^2 dA_\alpha(w)\Big)^{\frac{1}{2}} \Big(\int_{D_m} |g(w)|^2 dA_\alpha(w)\Big)^{\frac{1}{2}}\Big]^2$$

$$\leq \Big(\sum_m \int_{D_m} |f(w) - f(a_m)|^2 dA_\alpha(w)\Big) \Big(\sum_m \int_{D_m} |g(w)|^2 dA_\alpha(w)\Big)$$

$$\leq C_1 s^2 \sum_m \int_{D(a_m, r^{\frac{1}{4}})} |f(w)|^2 dA_\alpha(w) \frac{C}{r^{\frac{1}{2}}} ||g||_{2,\alpha}^2$$

$$\leq C_1 C^2 \frac{s^2}{r} ||f||_{2,\alpha}^2 ||g||_{2,\alpha}^2$$

and $\Big(\sum_m \int_{D_m} f(a_m)\overline{(g(w) - g(a_m))} dA_\alpha(w)\Big)^2$

$$\leq C_1 C^2 \frac{s^2}{r} ||f||_{2,\alpha}^2 ||g||_{2,\alpha}^2,$$

we get

$$\begin{aligned} < (I - T_{\mu_r}^\alpha)(f), g > \ &\leq \ |\sum_m \int_{D_m} (f(w) - f(a_m))\overline{g(w)} dA_\alpha(w)| \\ &\quad + |\sum_m \int_{D_m} f(a_m)\overline{(g(w) - g(a_m))} dA_\alpha(w)| \\ &\leq \ 2C_1^{\frac{1}{2}} C \frac{s}{r^{\frac{1}{2}}} ||f||_{2,\alpha} ||g||_{2,\alpha}. \end{aligned}$$

Since $\lim_{r\to 0} \frac{s}{r} = 1$, $\lim_{r\to 0} ||I - T_{\mu_r}^\alpha|| = 0$. □

Consider $D(a, r) \cap \{a_1, a_2, \cdots\}$. We notice that $\beta(a_i, a_j) \geq \frac{r}{2}$ for $i \neq j$. Since $\lim_{|a|\to 1^-} \frac{1 - |a|^2}{1 - s^2|a|^2} s = 0$, $\lim_{|a|\to 1^-} \frac{\mu_r(D(a, r))}{(1 - |a|^2)^{2+\alpha}} = 0$ whenever $\alpha < 0$ and hence μ_r is a vanishing Carleson measure. Thus $T_{\mu_r}^\alpha$ is a compact operator.

PROPOSITION 2.5. *If there is $t \in (0, 1)$ such that $\{a_1, a_2, \cdots\} \subseteq B(0, t)$ then $T_{\mu_r}^\alpha$ is a compact operator.*

Proof. It follows immediately from the fact that $\mu_r(\mathbb{D}\setminus B(0,t)) = 0$. □

COROLLARY 2.6. *If $dist(\{a_1, a_2, \cdots\}, \partial\mathbb{D}) > 0$ then $T^{\alpha}_{\mu_r}$ is a compact operator.*

Let $\mu^{\alpha}_r = \sum_{m=1}^{\infty} A_{\alpha}(D_m)\delta_{\{a_m\}}$, where $\delta_{\{a\}}$ is simply the unit atomic measure at $\{a\}$. Then $\mu^0_r = \mu_r$ and $A_{\alpha}(D_k) \approx (1-|a_k|^2)^{2+\alpha}$. Since $D\left(a_k, \frac{r}{4}\right) \subset D_k \subset D(a_k, r)$, for $z \in D_k$, $(1-|z|^2)^{\alpha} \approx (1-|a_k|^2)^{\alpha}$. We notice that $\{D_k\}$ is a disjoint family and $\bigcup D_k = \mathbb{D}$ and hence $\sum_k A_{\alpha}(D_k) = A_{\alpha}(\mathbb{D}) < \infty$, that is, $\sum_k (1-|a_k|^2)^{2+\alpha} < \infty$.
Suppose $f \in A^2_{\alpha}$. By Proposition 4.13 ([1]), there is a constant C such that

$$|f(a_k)|^2 \leq \frac{C}{(1-|a_k|^2)^{2+\alpha}} \int_{D_m} |f(w)|^2 dA_{\alpha}(w).$$

If $\inf\{A_{\alpha}(D_m) : m \in \mathbb{N}\} > 0$ then $\{f(a_k)\}$ is in l^2. Using the exactly same arguments for a measure μ_r, μ^{α}_r is a Carleson measure and hence $T^{\alpha}_{\mu^{\alpha}_r}$ is a bounded Toeplitz operator.

Since $T^{\alpha}_{\mu^{\alpha}_r}(f)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha} f(w)}{(1-z\overline{w})^{2+\alpha}} d\mu^{\alpha}_r(w)$,

$$\begin{aligned}(T^{\alpha}_{\mu^{\alpha}_r})^{(n)}(f)(z) &= (-1)^n(\alpha+1)(\alpha+2)\cdots(\alpha+n+1) \\ &\quad \times \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha} f(w)(\overline{w})^n}{(1-z\overline{w})^{2+n+\alpha}} d\mu^{\alpha}_r(w).\end{aligned}$$

Since μ^{α}_r is a Carleson measure for A^1_{α},

$$\begin{aligned}|(T^{\alpha}_{\mu^{\alpha}_r})^{(n)}(f)(z)| &\leq \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+1)} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha}|f(w)|}{|1-z\overline{w}|^{2+n+\alpha}} dA_{\alpha}(w) \\ &\leq \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+1)} \frac{1}{(1-|z|)^{2+n+\alpha}} \\ &\quad \times \left(\int_{\mathbb{D}} (1-|w|^2)^{2\alpha} dA_{\alpha}(w)\right)^{\frac{1}{2}} ||f||_{2,\alpha}.\end{aligned}$$

If $\alpha > -\frac{1}{3}$ then one has the inequality whenever $f \in A^2_{\alpha}$.

References

[1] S. Axler, *Bergman Spaces and Their operators,* Pitman Research Notes in Mathematics, vol. 171, 1-50, Longman, Harlow.

[2] S. Axler and D. Zheng, *Compact operators via the Berezin Transform,* Indiana Univ. Math. **47** (1988), 387-399.

[3] K. H. Zhu, *Hankel-Toeplitz type operators on* $L_a^1(\Omega)$, Integral Equations and operator Theory **13** (1990), 285-302.

[4] K. Zhu, *Operator Theory in Function Spaces, 2ed.*, American Mathematical Society, 2007.

*

Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Republic of Korea
E-mail: shkang@sookmyung.ac.kr